On the scaling properties of (2+1) directed polymers in the high temperature limit

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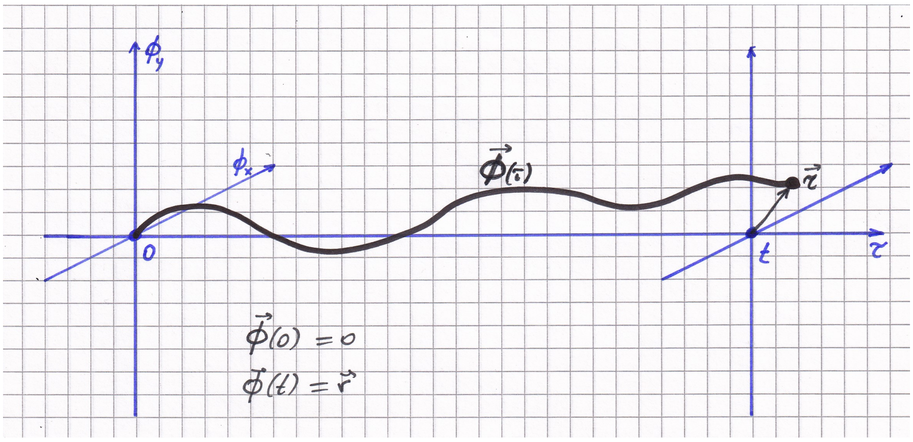
The model

The elastic string $\boldsymbol{\phi}(\tau)$ is described by the two-dimensional vector $\big(\phi_x(\tau),\, \phi_y(\tau)\big)$



The Hamiltonian:

$$H[\boldsymbol{\phi}(\tau); V] = \int_0^t d\tau \left\{ \frac{1}{2} \big[\partial_\tau \boldsymbol{\phi}(\tau)\big]^2 + V[\boldsymbol{\phi}(\tau), \tau] \right\}; \tag{1}$$

The disorder potential $V[\boldsymbol{\phi}, \tau]$ is Gaussian distributed:

$$\overline{V(\boldsymbol{\phi}, \tau) V(\boldsymbol{\phi}', \tau')} = u\, \delta(\tau - \tau') U(\boldsymbol{\phi} - \boldsymbol{\phi}') \tag{2}$$

$$U(\boldsymbol{\phi}) \;=\; \frac{1}{2\pi\, \epsilon^2}\; \exp\left\{ -\frac{\boldsymbol{\phi}^2}{2\epsilon^2} \right\} \tag{3}$$

The partition function:

$$Z(\mathbf{r}, t) = \int_{\phi(0)=\mathbf{0}}^{\phi(t)=\mathbf{r}} \mathcal{D}\boldsymbol{\phi}(\tau) \exp\big\{ -\beta H[\boldsymbol{\phi}(\tau), V] \big\} \;=\; \exp\big\{ -\beta F(\mathbf{r}, t) \big\} \tag{4}$$

where $F(\mathbf{r}, t)$ is the free energy which is a random quantity

• In one-dimensional model the fluctuations of the free energy are described by the Tracy-Widom (TW) distribution and their typical value scale with time as $t^{1/3}$ at all temperatures.

• In the considered (2+1) model due to extensive numerical simulations it is convincingly established that at the *zero-temperature* the free energy fluctuations scale as t^{θ} with the scaling exponent $\theta \simeq 0.241$.

• Here I would like to propose an approximate method which in the *high temperature limit* allows to derive the *left tail* asymptotics of the free energy distribution function. Assuming that this distribution function is defined by the only energy scale one finds that the scaling exponent $\theta = 1/2$, which implies that θ is non-universal being temperature dependent.

The free energy probability distribution function can be studied in terms of the integer moments of the partition function:

$$\overline{Z^N} \equiv Z(N,t) \; = \; \int_{-\infty}^{+\infty} dF \, P(F) \, \exp\{-\beta N F\} \tag{5}$$

where

$$Z(N,t) \; = \; \prod_{a=1}^{N} \int_{\boldsymbol{\phi}_a(0)=0}^{\boldsymbol{\phi}_a(t)=0} \mathcal{D}\boldsymbol{\phi}_a(\tau) \; \exp\Big\{-\beta H_N[\boldsymbol{\phi}_1(\tau), \, \boldsymbol{\phi}_2(\tau), \, \dots, \boldsymbol{\phi}_N(\tau)]\Big\} \tag{6}$$

is the replica partition function and

$$\beta H_N = \int_0^t d\tau \left[\frac{1}{2}\beta \sum_{a=1}^{N} \Big(\partial_\tau \boldsymbol{\phi}_a(\tau)\Big)^2 \; - \; \frac{1}{2}\beta^2 \, u \sum_{a,b=1}^{N} U\big(\boldsymbol{\phi}_a(\tau) - \boldsymbol{\phi}_b(\tau)\big)\right]; \tag{7}$$

is the replica Hamiltonian which describes N elastic strings $\big\{\boldsymbol{\phi}_1(\tau), \, \boldsymbol{\phi}_2(\tau), \, \dots, \boldsymbol{\phi}_N(\tau)\big\}$ with the attractive interactions $U\big(\boldsymbol{\phi}_a - \boldsymbol{\phi}_b\big)$.

To compute $Z(N,t)$ one introduces the function:

$$\Psi(\mathbf{r}_1, \, \mathbf{r}_2, \, \dots \mathbf{r}_N; \; t) \; = \; \prod_{a=1}^{N} \int_{\boldsymbol{\phi}_a(0)=\mathbf{0}}^{\boldsymbol{\phi}_a(t)=\mathbf{r}_a} \mathcal{D}\boldsymbol{\phi}_a(\tau) \; \exp\Big\{-\beta H_N\Big\} \tag{8}$$

such that $Z(N,t) \; = \; \Psi(\mathbf{r}_1, \, \mathbf{r}_2, \, \dots \mathbf{r}_N; \, t)\big|_{\mathbf{r}_a=0}$

$\Psi(\mathbf{r}_1, \, \mathbf{r}_2, \, \dots \mathbf{r}_N; \, t)$ is defined by the imaginary time Schrödinger equation

$$\beta \frac{\partial}{\partial t}\Psi \; = \; \frac{1}{2}\sum_{a=1}^{N} \Delta_a \Psi \; + \; \frac{1}{2}\beta^3 u \sum_{a,b=1}^{N} U(\mathbf{r}_a - \mathbf{r}_b) \, \Psi \tag{9}$$

The corresponding eigenvalue equation for the eigenfunctions $\psi(\mathbf{r}_1, \, \mathbf{r}_2, \, \dots \mathbf{r}_N)$, defined by the relation

$$\Psi(\mathbf{r}_1, \, \mathbf{r}_2, \, \dots \mathbf{r}_N; \; t) \; = \; \psi(\mathbf{r}_1, \, \mathbf{r}_2, \, \dots \mathbf{r}_N) \, \exp\big\{-t \, E_N\big\} \tag{10}$$

reads:

$$-2\beta \, E_N \, \psi \; = \; \sum_{a=1}^{N} \Delta_a \psi \; + \; \beta^3 u \sum_{a,b=1}^{N} U(\mathbf{r}_a - \mathbf{r}_b) \, \psi \tag{11}$$

• In the one-dimensional case the solution of the above equation is given by the Bethe ansatz wave function which is valid only for $U(x) = \delta(x)$ and which is based on the exact two-particle wave functions ($N = 2$) solution exhibiting finite value energy $E_{N=2}$.

• I contrast to that, in two dimensions there exists no *finite* two-particle solution for $U(\mathbf{r}) = \delta(\mathbf{r})$. In the limit $\epsilon \to 0$ (when $U(\mathbf{r})$ turns into the δ-function) the ground state energy $E_{N=2} \to -\infty$. In other words, in two dimensions we have to study the system with *finite* size function $U(\mathbf{r})$ and the value of its spatial size ϵ must explicitly enter into the final results.

• Besides, in two dimensions even the ground state energy E_N as well as N-particle ground state wave function $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots \mathbf{r}_N)$ are not known.

• Here we propose un approach which makes possible to estimate the replica partition function $Z(N, t)$ in the limit $N \gg 1$. This in turn, at least in the high-temperature limit, allows to derive the scaling exponent of the free energy fluctuations.

Mean Field Approach

In the limit of large number of particles, $N \gg 1$, one can use the mean field approximation, in which the N-particle wave function factorizes into the product of N one-particle functions:

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) \simeq \prod_{a=1}^{N} \psi(\mathbf{r}_a) \tag{12}$$

In the leading order in N^{-1} one gets:

$$\Delta\psi(\mathbf{r}) - \lambda\,\psi(\mathbf{r}) + \kappa\,\psi(\mathbf{r}) \int d^2r'\, U_0(\mathbf{r}-\mathbf{r}')\,\psi(\mathbf{r}') = 0 \tag{13}$$

where

$$\lambda = -\frac{4\beta\epsilon^2}{N} E_N \tag{14}$$

$$\kappa = 2\beta^3 u\, N \tag{15}$$

and

$$U_0(\mathbf{r}) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\mathbf{r}^2\right\} \tag{16}$$

$$\int d^2r\, \psi(\mathbf{r}) = 1 \tag{17}$$

- Further strategy:

(1) For given values of the parameters λ and κ we have to find smooth non-negative solution of eq.(13) such that $\psi(\mathbf{r} \to \infty) \to 0$.

(2) Substituting this solution into the constraint (17) we can find λ as a function of κ, which eventually gives us the dependence of the ground state energy E_N on the replica parameter N.

The example of $(1+1)$ system:

The one-dimensional version of eqs.(13)-(17) reads

$$\psi''(x) \; - \; \lambda\,\psi(x) \; + \; \kappa\,\psi(x) \int dx'\, U_1(x-x')\,\psi(x') \; = \; 0 \tag{18}$$

$$\int_{-\infty}^{+\infty} dx\, \psi(x) \; = \; 1 \tag{19}$$

$$U_1(x) \; = \; \frac{1}{\sqrt{2\pi}} \; \exp\biggl\{-\frac{1}{2}\,x^2\biggr\} \tag{20}$$

Redefining

$$\psi(x) \; = \; \frac{\lambda}{\kappa}\,\phi\bigl(\sqrt{\lambda}\,x\bigr) \tag{21}$$

we get

$$\phi''(z) \; - \; \phi(z) \; + \; \phi(z) \int dz'\, U_\lambda(z-z')\,\phi(z') \; = \; 0 \tag{22}$$

$$\frac{\sqrt{\lambda}}{\kappa} \int_{-\infty}^{+\infty} dz\, \phi(z) \; = \; 1 \tag{23}$$

$$U_\lambda(z) \; = \; \frac{1}{\sqrt{2\pi\lambda}} \; \exp\biggl\{-\frac{1}{2\lambda}\,x^2\biggr\} \tag{24}$$

According to eq.(23),

$$\lambda \; = \; \biggl(\int_{-\infty}^{+\infty} dz\, \phi(z)\biggr)^{-2} \kappa^2 \tag{25}$$

In the high temperature limit both $\kappa \propto \beta^3 uN \to 0$ and $\lambda \to 0$, so that

$$\lim_{\beta \to 0} \; U_\lambda(z) \; \to \; \delta(z) \tag{26}$$

and eq.(22) reduces to

$$\phi''(z) \; - \; \phi(z) \; + \; \phi^2(z) \; = \; 0 \tag{27}$$

This equation has an instanton-like solution with $\phi(0) \simeq 1.50$, $\phi'(0) = 0$ and $\phi(z \to \infty) \to 0$:

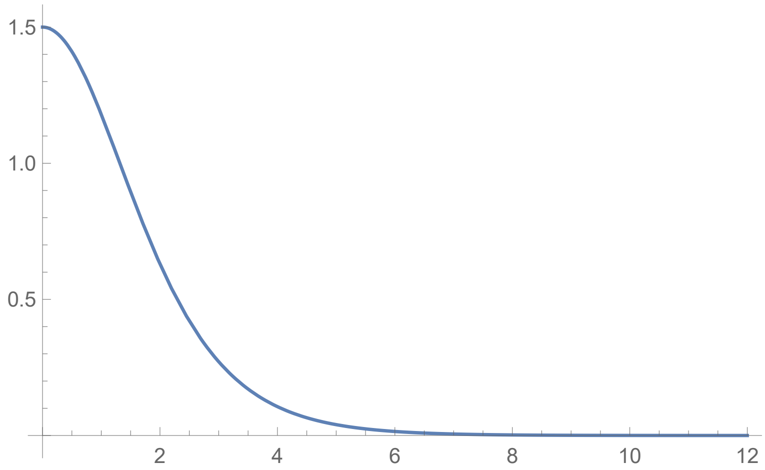


FIG. 1: Instanton solution of eq.(27)

According to eqs.(14), (15) and (25) we find

$$E_N \simeq -\frac{1}{36}\beta^5 u^2 N^3 \propto -N^3 \quad (28)$$

This result, except for the numerical prefactor, perfectly fits with the exact value of ground state energy $-\frac{1}{24}\beta^5 u^2 N^3$ of the one-dimensional N-particle boson system and correspondingly provide the well known value of the free energy scaling exponent $\theta = 1/3$.

(2+1) directed polymers

For the radially symmetric wave function $\psi(\mathbf{r}) = \psi(|\mathbf{r}|) \equiv \psi(r)$. eqs. (13)-(17) take the form

$$\psi''(r) \; + \; \frac{1}{r}\psi'(r) \; - \; \lambda\psi(r) \; + \; \kappa\,\psi(r) \int d^2r'\, U_0(|\mathbf{r}-\mathbf{r}'|)\, \psi(r') \; = \; 0 \tag{29}$$

$$2\pi \int_0^{\infty} dr\, r\, \psi(r) \; = \; 1 \tag{30}$$

Redefining

$$\psi(r) \; = \; \frac{\lambda}{\kappa}\, \phi\big(\sqrt{\lambda}\, r\big) \tag{31}$$

we get

$$\phi''(z) \; + \; \frac{1}{z}\phi'(z) \; - \; \phi(z) \; + \; \phi(z) \int d^2z'\, U_\lambda(|\mathbf{z}-\mathbf{z}'|)\, \phi(z') \; = \; 0 \tag{32}$$

$$2\pi \int_0^{+\infty} dz\, z\, \phi(z) \; = \; \kappa \tag{33}$$

$$U_\lambda(|\mathbf{z}|) \; = \; \frac{1}{2\pi\lambda} \; \exp\Big\{ -\frac{1}{2\lambda}\, |\mathbf{z}|^2 \Big\} \tag{34}$$

In the high temperature limit

$$\lim_{\lambda\to 0} U_\lambda(z) \; = \; \delta(z) \tag{35}$$

eq.(32) reduces to

$$\phi''(z) \; + \; \frac{1}{z}\phi'(z) \; - \; \phi(z) \; + \; \phi^2(z) \; = \; 0 \tag{36}$$

This equation has an instanton-like solution with $\phi(0) \simeq 2.39$, $\phi'(0) = 0$ and $\phi(z \to \infty) \to 0$:

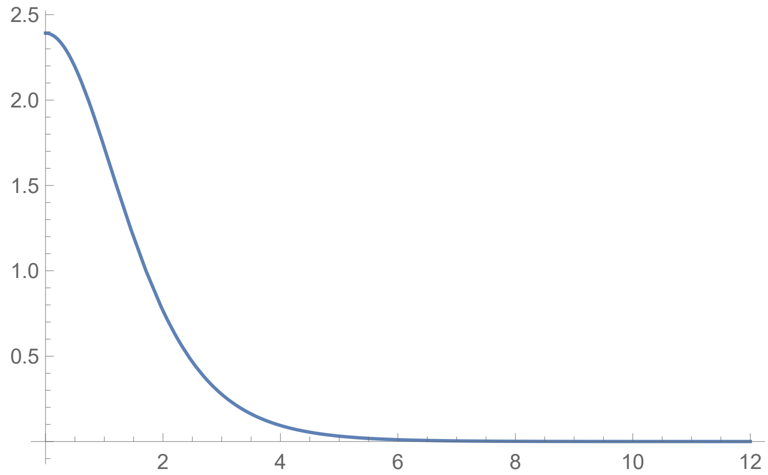


FIG. 2: Instanton solution of eq.(36)

Substituting this solution into eq.(33) we find

$$\kappa(\lambda = 0) \;\equiv\; \kappa_0 \;\simeq\; 31.00 \tag{37}$$

At non-zero $\lambda \ll 1$, numerical solution of eqs.(32)-(33) demonstrate perfect linear dependence:

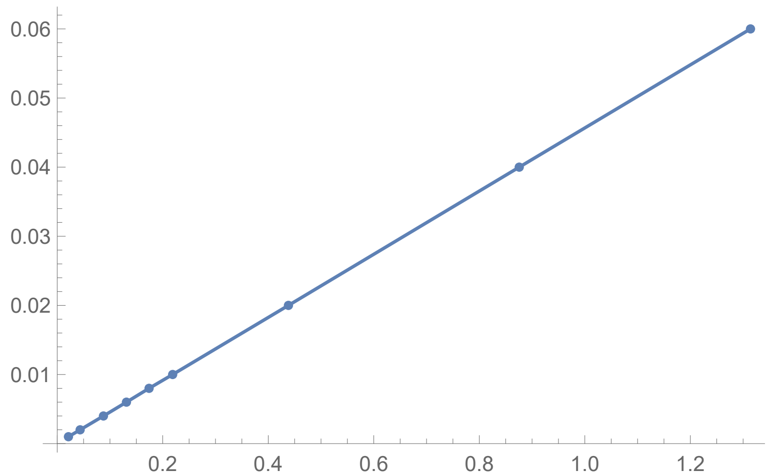


FIG. 3: Dependence of λ on $(\kappa - \kappa_0)$

$$\lambda(\kappa) \;=\; \gamma\,(\kappa - \kappa_0) \tag{38}$$

with

$$\gamma \;\simeq\; 0.050 \tag{39}$$

Free energy scaling

For the ground state energy we find

$$E_N \;\simeq\; -\frac{\gamma}{4\beta\epsilon^2}\, N\left(2\beta^3 uN \;-\; \kappa_0\right) \tag{40}$$

which is valid for

$$N \;>\; N_* \equiv \frac{\kappa_0}{2\beta^3 u} \;\gg\; 1 \tag{41}$$

For the replica partition function we obtain

$$Z(N,t) \;\sim\; \exp\left\{\frac{\gamma}{4\beta\epsilon^2}\, N\left(2\beta^3 uN \;-\; \kappa_0\right) t\right\} \tag{42}$$

Correspondingly, at large N we have

$$\int_{-\infty}^{+\infty} dF\, P(F)\, \exp\{-\beta N F\} \;\sim\; \exp\left\{\frac{\gamma u}{2\epsilon^2}\,(\beta N)^2\, t \;-\; \frac{\gamma\kappa_0}{4\beta^2\epsilon^2}\,\beta N\, t\right\} \tag{43}$$

The total free energy F splits into two *independent* parts: $F \;=\; \overline{F} \;+\; \tilde{F}$, where $\overline{F} \;=\; \gamma\kappa_0 t/(4\beta^2\epsilon^2)$ is an extensive non-random (selfaveraging) part while $\tilde{F}$ is the fluctuating contribution described by a distribution function $\tilde{P}\big(\tilde{F}\big)$ which is defined by the relation

$$\int_{-\infty}^{+\infty} d\tilde{F}\, \tilde{P}\big(\tilde{F}\big)\, \exp\{-\beta N\tilde{F}\} \;\sim\; \exp\left\{\frac{\gamma u}{2\epsilon^2}\, t\,(\beta N)^2\right\} \tag{44}$$

As eq.(44) is valid only for $N \;>\; N_* \;\gg\; 1$, the above equation gives us only the *left* tail of this distribution:

$$\tilde{P}\big(\tilde{F} \to -\infty\big) \;\sim\; \exp\left\{-\frac{\epsilon^2}{2\gamma\, u\, t}\,\tilde{F}^2\right\} \tag{45}$$

Thus, the typical value of the free energy fluctuations scale as

$$\tilde{F} \;\sim\; \frac{\sqrt{\gamma\, u}}{\epsilon}\, t^{1/2} \tag{46}$$

Conclusions:

- The fact that at high temperatures the scaling exponent ($\theta = 1/2$) is different from the one at the zero temperature ($\theta \simeq 0.241$) indicates that in two dimensions this scaling exponent must be temperature dependent.

- In the high temperature limit the prefactor in time scaling of the free energy fluctuations $\sim \frac{\sqrt{u}}{\epsilon} t^{1/2}$ is defined by the parameters of the disorder potential and it is temperature independent (unlike (1+1) case, where it is proportional to $\beta^{2/3}$).

- The results presented above are based on two crucial assumptions:

  (1) pure heuristic mean-field ansatz for the N-particle wave function, $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots \mathbf{r}_N) \;\rightarrow\; \psi(\mathbf{r}_1)\,\psi(\mathbf{r}_2)\dots\psi(\mathbf{r}_N)$;

  (2) the hypothesis that the entire free energy probability distribution function $\tilde{P}(\tilde{F})$ can be reduced to a *universal* function.